A Competitive Game for Joint Replenishment

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We consider the jointly replenishing N identical firms that operate under an EOQ like setting using a non-cooperative game. In this game, each firm bids how much it is willing to pay per unit of its replenishment and a service provider determines the frequency of the joint orders. We characterize the equilibrium under full and asymmetric information.

Keywords: Joint replenishment, Economic Order Quantity model, Non–cooperative game theory

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1. Introduction

One of the most fundamental trade–offs in operations is between cycle stocks and setup costs associated with production, transportation or procurement. Larger lot sizes lead to larger cycle stocks, while smaller lot sizes result in more orders over time and thus more setup costs. These trade–offs are studied formally since 1913 starting with the classical Economic Order Quantity (EOQ) model [3]. In EOQ model, a firm faces a constant and deterministic demand of λ per unit of time, pays a setup cost of K for each replenishment order and incurs an inventory holding cost of h per unit of inventory it keeps per unit of time. Minimizing setup costs and inventory holding costs leads to the optimal order quantity $Q^* = \sqrt{2K\lambda/h}$, at which inventory holding and setup costs are equal to each other. Since then, there has been a vast amount of literature on lot sizing that relaxes the certain restrictive assumptions of the EOQ model. The interested reader is referred to [5] for a recent review of research in this area.

A major cost saving opportunity in this setting is joint replenishment, i.e., consolidating orders for different items (or locations). By carefully coordinating the replenishment of multiple items, one can exploit the economies of scale of ordering jointly and reduce setup costs, cycle inventories or both. Finding a joint replenishment policy to minimize the total setup costs and inventory costs is known as the Joint Replenishment Problem in the literature. There is also large body of research in this area: see [6] for a recent and [1] for an earlier review.

Although joint replenishment may be a significant means to reduce costs, when it involves a group of items or locations that are not controlled centrally, it is not always apparent how to split these savings among the parties fairly. A fair allocation is necessary to induce different decentralized entities to engage in cooperation. Even if the items or locations are managed by a single firm, a fair allocation is still important from a cost accounting perspective. Recently, cooperative game theory models are developed to investigate whether a fair allocation of total savings (or total costs) is possible and if so, how. In the first of these models, Meca et al. [8] show that it is possible to coordinate the system (obtain minimum total cost) when the players only share their order frequencies prior to joint replenishment. They propose an allocation mechanism which distributes the total setup cost among the jointly replenished locations in proportion to the square of their order frequencies and show that this allocation is in the core of the game, i.e., the firms cannot decrease their costs further by defecting from the grand coalition of firms. In [8], there are only major setup costs, i.e., setup costs are independent of what items are included in the order. When there are also minor setup costs associated with each item, it is not always optimal to order every item with every replenishment. In fact, the structure of the policy that minimizes the total costs is not known. For this problem, Hartman and Dror [4] show that the game with a specific group of items has a core, whenever these items need to be ordered together on the same schedule to minimize total costs. Anily and Haviv [2] limit their attention to the near optimal power–of–two policies for this problem, and show the existence and example of a core allocation of total costs.

In this paper, we take a non–cooperative approach to the joint replenishment problem. To our knowledge, [7] is the only study in the literature that follows this approach. In [7], the allocation of total costs is done by the rule in [8], but it is assumed that order frequencies when the firms order independently are observable but not verifiable. Therefore, each firm can report an order frequency that is different from its true order frequency. The joint order frequency is determined based on the reported order frequencies and joint total costs are incurred accordingly. It is shown that there exists a Nash equilibrium under which each firm participates in the joint replenishment program under certain conditions, but this equilibrium leads to higher costs than cooperative solution for each firm when there are more than 3 firms in the game.

We use a more direct approach in our paper. We assume that each firm bids how much it is willing to pay for each unit of its replenishment to the provider of the service, which we call the "replenishment service provider" or RSP. The setups may be due to transportation, production or procurement leading to different services offered by this firm. The RSP only has the objective of ensuring a revenue of K from each replenishment. Therefore, he sets a joint cycle length just enough to obtain K for each replenishment. Since each firm's inventory holding costs depend on this cycle length and this cycle length is a function of individual bids, this is a competitive game in which each firm's strategy is its bid per unit of its replenishment. Our model is different from [7] in a number of important ways. First, we use a simple, direct, and a truthful mechanism to split the ordering cost between different firms. Each firm bids how much it is willing to pay rather than bidding untruthfully its order frequency when it orders independently. Finally, the allocation of the order cost in our game depends directly on the bids placed by the firms rather than an allocation that is designed for a cooperative solution.

Our first model assumes that we have N identical firms with equal demand rates and inventory holding cost rates, which are publicly known by all parties in the game. Our solution concept in this game is Nash equilibrium. It is shown that, in equilibrium, each firm pays $1/N$ of what it pays when ordering individually per unit of its replenishment. This leads to an order cycle length (or frequency) which is exactly same as when ordering individually. Thus while the total setup costs are reduced to $1/N$ of when ordering individually, the total inventory holding costs remain the same. The resulting total costs are $(N+1)/2N$ of total costs under independent ordering. The cooperative solution, on the other hand, can achieve a total cost of $1/\sqrt{N}$ of the total cost under independent ordering, since it also creates a reduction in cycle length and thus a reduction in inventory holding costs as well as setup costs.

In the second model, we introduce private information regarding demand rates. It is assumed that each firm's demand rate can take one of two values (types). Each firm learns its type prior to bidding, but does not reveal this information to other firms. Our solution concept in this case is Bayesian Nash equilibrium. A Bayesian Nash equilibrium is a Nash equilibrium where each player, given his type, selects a best response against the average best responses of the competing players. The conditions for the Bayesian Nash equilibrium are derived. In this case, the gain from the bidding game is due to the fact that more information about the demand rates is making its way to the joint replenishment decisions of the RSP. A numerical study is conducted to show that the performance of the competitive solution behaves similar to the case of full information as N increases, but information asymmetry tends to offer improvements as N and variability in demand rates increase.

The rest of this paper is organized as follows. In Section 2, we model the competitive game for the case of full information and derive the equilibrium conditions along with a comparison of competitive solution with independent and cooperative ordering. In Section 3, we model the competitive game under private information, derive the equilibrium conditions and report the findings of a numerical study that compares the three solutions. Section 4 concludes the paper along with avenues for future research.

2. Full Information

Assume that we have N identical firms. Each firm is facing a constant deterministic demand with rate λ per unit of time. Inventory holding cost rate is h per unit per unit of time. The firms order their goods from a company which we call the replenishment service provider (RSP). The RSP, for example, may be a transportation service provider, if the setups are due to transportation, or a manufacturing company if the setups are due to switchovers in manufacturing. The RSP charges K per order regardless of how much is shipped. For simplicity, assume that the variable cost is zero.

When the firms operate in a decentralized fashion and order independently, each firm has the total cost function

$$\frac{K\lambda}{Q} + \frac{1}{2} h Q.$$

It is well known that each firm's optimal order quantity and optimal cycle length are

$$Q^d = \sqrt{\frac{2K\lambda}{h}}, \text{ and } T^d = \sqrt{\frac{2K}{\lambda h}},$$

which lead to a per unit replenishment cost of

$$p^d = \frac{K}{Q^d} = \sqrt{\frac{Kh}{2\lambda}}.$$

Each firm's cost per unit of time is $\sqrt{2K\lambda h}$. Thus, the total cost for N firms in this case is

$$TC^d = N\sqrt{2K\lambda h}.$$

When the firms fully cooperate and order jointly under a centralized scheme, the firms have a common cycle length. The total costs for N firms can be written as

$$TC(T) = \frac{K}{T} + \frac{1}{2} N \lambda h T.$$

The optimal cycle length can be easily found as

$$T^c = \frac{1}{\sqrt{N}}\sqrt{\frac{2K}{\lambda h}}.$$

The optimal total cost for the cooperative case is then

$$TC^c = \sqrt{N}\sqrt{2K\lambda h}.$$

At each cycle, each firm orders

$$Q^c = \lambda\, T^c = \frac{1}{\sqrt{N}}\sqrt{\frac{2K\lambda}{h}},$$

and pays $1/N$ of the order cost K. This leads to a per unit replenishment cost of

$$p^c = \frac{K/N}{Q^c} = \frac{1}{\sqrt{N}}\sqrt{\frac{Kh}{2\lambda}}.$$

The benefit of joint replenishment in this setting is obvious. The total costs (and each firm's cost) is reduced to $1/\sqrt{N}$ of costs when firms order independently. Cycle length, each firms order quantity and per unit replenishment costs are also $1/\sqrt{N}$ of the corresponding values in independent ordering. With joint replenishment, firms share the order costs and they are able to order more frequently.

We now study the competitive case as cooperation may not always be possible between these N firms. For this case, we propose the following mechanism. Each firm bids a replenishment cost per unit that they are willing to pay under joint replenishment. Based on these bids and demand rates, the RSP determines the cycle length which will ensure a revenue of K for him. We assume that the RSP is not a profit maximizer and he only aims to gain K for each replenishment, which may be equal to his own costs plus a margin as before. Let $p_1, p_2, ..., p_N$ be the bids that are placed by firms. Then the cycle length that is determined by the RSP will be

$$T = \frac{K}{\lambda \sum_{k=1}^{N} p_k}.$$

The costs per unit of time for firm j that bids a price of p_j can be written as

$$\phi_j(p_j, \mathbf{p_{-j}}) = \frac{1}{2}\, h\, \lambda\, T + p_j\, \lambda = \frac{h\, K}{2\sum_{k=1}^{N} p_k} + p_j\, \lambda \tag{1}$$

where $\mathbf{p_{-j}}$ is a vector of bids except the bid of firm j.

Next theorem characterizes the Nash equilibrium for this game.

Theorem 1. *Any vector of bids $\mathbf{p} = (p_1, p_2, ..., p_N)$ that satisfies (2) is a Nash equilibrium.*

$$\sum_{k=1}^{N} p_k = \sqrt{\frac{Kh}{2\lambda}}. \tag{2}$$

Proof: Taking the derivative of (1) with respect to p_j

$$\frac{\partial \phi_j(p_j, \mathbf{p_{-j}})}{\partial p_j} = -\frac{h\, K}{2(\sum_{k=1}^{N} p_k)^2} + \lambda.$$

Equating this to zero leads to the condition (2) for firm j. We have the same condition for all firms. It can be easily shown that each function ϕ_j is convex in p_j. Thus any vector of bids $\mathbf{p} = (p_1, p_2, ..., p_N)$ that satisfy (2) is a Nash equilibrium. $\square$

As Theorem 1 states, there is a continuum of equilibria. However, since firms are identical, we can assume that a plausible outcome of the game is a symmetric equilibrium. Such an approach is widely used in economics literature, at least as a starting point, see for example [9] and [10]. Nevertheless, even if we use one of the asymmetric equilibria as an outcome of the game, our results do not change if we limit our attention to "average" firm behavior and total costs under equilibrium. For the symmetric equilibrium, we have the following lemma.

LEMMA 1. *The symmetric equilibrium is*

$$p_j^g = p^g = \frac{1}{N}\sqrt{\frac{Kh}{2\lambda}}, \text{ for all } j.$$

Cycle length as a result is

$$T^g = \frac{K}{\lambda N p^g} = \sqrt{\frac{2K}{\lambda h}}.$$

At each cycle, each firm orders

$$Q^g = \lambda T^g = \sqrt{\frac{2K\lambda}{h}}.$$

The equilibrium total cost for the firms is

$$TC^g = \frac{1}{2} h N \lambda T^g + p^g N\lambda = \frac{1}{2} h N \lambda\sqrt{\frac{2K}{\lambda h}} + \frac{1}{N}\sqrt{\frac{Kh}{2\lambda}} N\lambda = \frac{N+1}{2}\sqrt{2K\lambda h}.$$

We highlight that, in equilibrium, firms order with the same frequency as when they order independently. Their benefit from joint replenishment is only due to the fact that they can share the order costs rather than bearing them individually. Each firm pays $1/N$ of what they pay in independent ordering per unit of their order. Also notice that each firm's inventory holding costs is $\sqrt{K\lambda h/2}$ in equilibrium. This is exactly equal to the joint order costs in the system.

Table 1 Full Information

	Independent	Cooperative	Competitive
Transportation Cost Per Unit	$\sqrt{\frac{Kh}{2\lambda}}$	$\frac{1}{\sqrt{N}}\sqrt{\frac{Kh}{2\lambda}}$	$\frac{1}{N}\sqrt{\frac{Kh}{2\lambda}}$
Cycle Length	$\sqrt{\frac{2K}{\lambda h}}$	$\frac{1}{\sqrt{N}}\sqrt{\frac{2K}{\lambda h}}$	$\sqrt{\frac{2K}{\lambda h}}$
Order Quantity	$\sqrt{\frac{2K\lambda}{h}}$	$\frac{1}{\sqrt{N}}\sqrt{\frac{2K\lambda}{h}}$	$\sqrt{\frac{2K\lambda}{h}}$
Total Cost	$N\sqrt{2K\lambda h}$	$\sqrt{N}\sqrt{2K\lambda h}$	$\frac{N+1}{2}\sqrt{2K\lambda h}$

The results for independent ordering, joint ordering with cooperation and competition are summarized in Table 1. As expected, the lowest total costs are obtained under joint replenishment with cooperation. The total costs are $1/\sqrt{N}$ of the total costs under independent ordering. Joint replenishment under competition also leads to savings against independent ordering, however the whole potential of joint replenishment is not exploited. The performance of the competitive solution can be measured as percentage of maximum savings that can be obtained through joint replenishment. We can define this measure as:

$$100 \times \frac{N-(N+1)/2}{N-\sqrt{N}} = 100 \times \frac{N-1}{2(N-\sqrt{N})}. \tag{3}$$

3. Asymmetric Information

We now turn our attention to the case of private information for joint replenishment. We assume that h is known and constant among N firms. We assume that each firm's demand rate can take one of two values: a high value λ_H with probability q and a low value λ_L with probability $(1-q)$ (A similar analysis can be carried out when the uncertainty is defined for h rather than λ). In the case of independent ordering, we assume that each firm learns its demand rate (type) prior to determining its order quantity. In the case of joint ordering with cooperation, we assume that demand rates of all firms are known prior to establishing the joint replenishment cycle length. In the case of joint ordering with competition, we assume that each firm learns its type prior to bidding, but does not share this information with other firms. However, the RSP has access to demand rate information of all firms, prior to determining the cycle length that will ensure a revenue of K per replenishment order.

With independent ordering, each firm sets its order quantity based on his type. Following the analysis in Section 2, a firm of type H has the optimal order quantity, cycle length, cost per unit of time, and replenishment cost per unit:

$$Q_H^d = \sqrt{\frac{2K\lambda_H}{h}},\ T_H^d = \sqrt{\frac{2K}{\lambda_H h}},\ C_H^d = \sqrt{2K\lambda_H h},\ \text{and}\ p_H^d = \sqrt{\frac{Kh}{2\lambda_H}}.$$

Corresponding values for a firm of type L are:

$$Q_L^d = \sqrt{\frac{2K\lambda_L}{h}},\ T_L^d = \sqrt{\frac{2K}{\lambda_L h}},\ C_L^d = \sqrt{2K\lambda_L h},\ \text{and}\ p_L^d = \sqrt{\frac{Kh}{2\lambda_L}}.$$

Let n be the number of firms that are type H. It is well known that n has a binomial distribution with N and q. Thus, the expected total cost for the case of independent ordering can be written as:

$$\begin{aligned} ETC^d &= \sum_{n=0}^{N} \binom{N}{n} q^n (1-q)^{N-n} \left[n\sqrt{2K\lambda_H h} + (N-n)\sqrt{2K\lambda_L h} \right] \\ &= N\left[q\sqrt{2K\lambda_H h} + (1-q)\sqrt{2K\lambda_L h} \right]. \end{aligned}$$

Similar derivations will lead to average optimal cycle length, average order quantity, and average replenishment cost per unit expressions similar to those given in Table 1 column 1 except that we now use a weighted average of corresponding expressions for high and low types.

With joint ordering under cooperation, the joint replenishment cycle length is decided after the types of all firms are observed. Given that there are n firms of type H, the total cost can be written as a function of T as follows:

$$TC(T,n) = \frac{K}{T} + \frac{1}{2}h\left[n\lambda_H + (N-n)\lambda_L\right]T.$$

The optimal cycle length can be easily found as:

$$T^c(n) = \sqrt{\frac{2K}{h\left[n\lambda_H + (N-n)\lambda_L\right]}}$$

This leads to the following total costs for a given n

$$TC^c(n) = \sqrt{2Kh\left[n\lambda_H + (N-n)\lambda_L\right]}.$$

Since n has a binomial distribution with N and q, we can write the expected cycle length and expected total cost as:

$$ET^c = \sum_{n=0}^{N} \binom{N}{n} q^n (1-q)^{N-n} \sqrt{\frac{2K}{h\left[n\lambda_H + (N-n)\lambda_L\right]}}, \text{ and}$$
$$ETC^c = \sum_{n=0}^{N} \binom{N}{n} q^n (1-q)^{N-n} \sqrt{2Kh\left[n\lambda_H + (N-n)\lambda_L\right]}.$$

Replenishment cost per unit for a given n is:

$$p^c(n) = \frac{K}{T^c(n)[n\lambda_H + (N-n)\lambda_L]} = \sqrt{\frac{Kh}{2\left[n\lambda_H + (N-n)\lambda_L\right]}}.$$

Then, the expected replenishment cost per unit can be calculated as

$$p^c = \sum_{n=0}^{N} \binom{N}{n} q^n (1-q)^{N-n} \sqrt{\frac{Kh}{2\left[n\lambda_H + (N-n)\lambda_L\right]}}.$$

We now study joint ordering with competition under asymmetric information. The sequence of events in this case is as follows. First each firm learns its demand rate (type). Then each firm bids the price per unit that it wants to pay for replenishment. Each firm then communicates its demand rate with the RSP. The RSP sets the cycle length of the joint order such that he obtains a revenue of K per trip. Finally, the firms incur their costs according to this cycle length.

Let p_{jH} is the price that firm j bids if its demand is low. Similarly define p_{jL}. Let $\mathbf{v} = (v_1, v_2, \ldots, v_N)$ represent a realization of demand rates such that $v_j = 1$ if the demand rate for firm j is λ_H and $v_j = 0$ if the demand rate for firm j is λ_L in realization $\mathbf{v}$. Then, for a given $\mathbf{v}$ the RSP will set the cycle length

$$T(\mathbf{v}) = \frac{K}{\lambda_H \sum_{k=1}^{N} v_k p_{kH} + \lambda_L \sum_{k=1}^{N} (1-v_k) p_{kL}}.$$

Consider a firm j with a high type. The expected payoff for this firm can be written as

$$\phi_{jH}(p_{jH},\mathbf{p_{-j}}) = \frac{1}{2}\,h\,\lambda_H \sum_{\mathbf{v}} c_{jH}(\mathbf{v})\,T(\mathbf{v}) + p_{jH}\,\lambda_H. \tag{4}$$

Similarly for a firm j with a low type, the expected payoff can be written as

$$\phi_{jL}(p_{jL},\mathbf{p_{-j}}) = \frac{1}{2}\,h\,\lambda_L \sum_{\mathbf{v}} c_{jL}(\mathbf{v})\,T(\mathbf{v}) + p_{jL}\,\lambda_L. \tag{5}$$

In (4) and (5), $c_{jH}(\mathbf{v})$ and $c_{jL}(\mathbf{v})$ inside the summations are the conditional probabilities of realization $\mathbf{v}$, given firm j has a high type and a low type, respectively. These conditional probabilities can be written as follows:

$$c_{jH}(\mathbf{v}) = \begin{cases} 0, & \text{if } v_j = 0, \\ q^{\sum_{k=1}^{N} v_k - 1}(1-q)^{N-\sum_{k=1}^{N} v_k}, & \text{if } v_j = 1, \end{cases}$$

$$c_{jL}(\mathbf{v}) = \begin{cases} 0, & \text{if } v_j = 1, \\ q^{\sum_{k=1}^{N} v_k}(1-q)^{N-\sum_{k=1}^{N} v_k - 1}, & \text{if } v_j = 0. \end{cases}$$

The next theorem characterizes the Bayesian Nash equilibrium for the asymmetric information game.

THEOREM 2. *Any two vectors of bids $\mathbf{p_H} = (p_{H1}, p_{H2}, ..., p_{HN})$ and $\mathbf{p_L} = (p_{L1}, p_{L2}, ..., p_{LN})$ that satisfy (6) and (7) are a Bayesian Nash equilibrium.*

$$\sum_{\mathbf{v}} \frac{c_{jH}(\mathbf{v})}{[\lambda_H \sum_{k=1}^{N} v_k p_{kH} + \lambda_L \sum_{k=1}^{N}(1-v_k) p_{kL}]^2} = \frac{2}{h\,K\lambda_H}, \quad \textit{for all } j, \tag{6}$$

$$\sum_{\mathbf{v}} \frac{c_{jL}(\mathbf{v})}{[\lambda_H \sum_{k=1}^{N} v_k p_{kH} + \lambda_L \sum_{k=1}^{N}(1-v_k) p_{kL}]^2} = \frac{2}{h\,K\lambda_L}, \quad \textit{for all } j. \tag{7}$$

Proof: Taking the derivatives of (4) and (5) with respect to p_{jH} and p_{jL} respectively and setting them to zero will lead to

$$\frac{\partial \phi_{jH}(p_{jH},\mathbf{p_{-j}})}{\partial p_{jH}} = -\sum_{\mathbf{v}} \frac{h\,K(\lambda_H)^2 c_{jH}(\mathbf{v})}{2\,[\lambda_H \sum_{k=1}^{N} v_k p_{kH} + \lambda_L \sum_{k=1}^{N}(1-v_k) p_{kL}]^2} + \lambda_H = 0,$$

$$\frac{\partial \phi_{jL}(p_{jL},\mathbf{p_{-j}})}{\partial p_{jL}} = -\sum_{\mathbf{v}} \frac{h\,K(\lambda_L)^2 c_{jL}(\mathbf{v})}{2\,[\lambda_H \sum_{k=1}^{N} v_k p_{kH} + \lambda_L \sum_{k=1}^{N}(1-v_k) p_{kL}]^2} + \lambda_L = 0,$$

Simplifying and reorganizing lead to the desired conditions. It can also be easily verified that the functions ϕ_{jH} and ϕ_{jL} are convex in p_{jH} and p_{jL} which completes the proof. $\square$

As in Section 2, we restrict ourselves to symmetric equilibrium, and use the following lemma.

LEMMA 2. *The symmetric Bayesian Nash equilibrium (p_H^g, p_L^g) satisfies the following*

$$\sum_{n=0}^{N-1} \binom{N-1}{n} q^n (1-q)^{N-n-1} \frac{1}{[(n+1)p_H^g \lambda_H + (N-n-1)p_L^g \lambda_L]^2} = \frac{2}{h\,K\lambda_H}, \tag{8}$$

$$\sum_{n=0}^{N-1} \binom{N-1}{n} q^n (1-q)^{N-n-1} \frac{1}{[n\,p_H^g \lambda_H + (N-n)p_L^g \lambda_L]^2} = \frac{2}{h\,K\lambda_L}. \tag{9}$$

Proof: Having $p_{jH} = p_H$ and $p_{jL} = p_L$ lead us to work with only the number of high types (n) among $N-1$ firms other than firm j, which has a binomial distribution with parameters $N-1$ and q. Taking the expectation in (4) and (5) over n, and taking the derivative with respect to p_H and p_L, we can write the first order conditions as:

$$-\sum_{n=0}^{N-1}\binom{N-1}{n}q^n(1-q)^{N-n-1}\frac{h\,K(\lambda_H)^2}{2[(n+1)p_H\lambda_H+(N-n-1)p_L\lambda_L]^2}+\lambda_H=0$$
$$-\sum_{n=0}^{N-1}\binom{N-1}{n}q^n(1-q)^{N-n-1}\frac{h\,K(\lambda_L)^2}{2[n\,p_H\lambda_H+(N-n)p_L\lambda_L]^2}+\lambda_L=0$$

Simplifying and reorganizing lead to the desired condition. □

Note that the last terms inside summations in the left hand sides of equations (8) and (9) are $1/K^2$ times the cycle length given one firm is a particular type and n of the remaining firms are high type. Multiplying both sides of (8) and (9) by $hK\lambda_H/2$ and $hK\lambda_L/2$, respectively, leads to an interesting interpretation. In equilibrium, each firm, given its type, bids such that the expected value of its own inventory holding cost divided by the joint order cost is equal to 1. This is an extension of a similar equilibrium behavior in the full information case, where each firm bids such that one firm's inventory holding costs are equal to the joint order costs.

For a given number of high types (among N firms), the cycle length is given as

$$T(n)=\frac{K}{p_H^g n\lambda_H+p_L^g(N-n)\lambda_L}.$$

Then, the total cost for all firms for a given number of high types can be written as

$$TC(n)=\frac{1}{2}\,h\,[n\lambda_H+(N-n)\lambda_L]\frac{K}{p_H^g n\lambda_H+p_L^g(N-n)\lambda_L}+p_H^g n\lambda_H+p_L^g(N-n)\lambda_L.$$

The expected cycle length and expected total cost for all firms are then given as follows

$$ET^g=\sum_{n=0}^{N}\binom{N}{n}q^n(1-q)^{N-n}T(n),\ \text{and}$$
$$ETC^g=\sum_{n=0}^{N}\binom{N}{n}q^n(1-q)^{N-n}TC(n),$$

since n has a binomial distribution with parameters N and q.

We conduct a limited computational study to see the impact of information asymmetry on the total costs. In Table 2, we provide the expected cycle lengths and expected total costs for different N and q obtained in three cases: independent ordering, joint ordering with cooperation, and joint ordering with competition. We also report the equilibrium bids when for the low and high type firms for the competition case. We measure the performance of the competitive solution as the percentage of maximum savings that can be obtained through joint replenishment, i.e.,

$$100\times\frac{ETC^d-ETC^g}{ETC^d-ETC^c}.$$

Table 2 **Asymmetric Information:** $\lambda_H = 20,\ \lambda_L = 10,\ K = 80,\ h = 1$

		Independent		Cooperative		Competitive				
q	N	ET^d	ETC^d	ET^c	ETC^c	p_L^g	p_H^g	ET^g	ETC^g	% Perf
0.25	2	3.71	88.28	2.58	62.80	0.9335	0.8728	3.67	66.49	85.51
0.25	3	3.71	132.43	2.10	77.09	0.5797	0.6878	3.64	88.45	79.46
0.25	4	3.71	176.57	1.81	89.12	0.4040	0.5920	3.61	110.03	76.09
0.25	5	3.71	220.71	1.61	99.71	0.2995	0.5318	3.58	131.23	73.95
0.25	10	3.71	441.42	1.14	141.21	0.0979	0.3921	3.43	232.22	69.69
0.50	2	3.41	96.57	2.36	68.78	0.8558	0.8273	3.37	72.78	85.63
0.50	3	3.41	144.85	1.91	84.45	0.4814	0.6253	3.32	96.62	79.86
0.50	4	3.41	193.14	1.65	97.63	0.2996	0.5194	3.28	119.85	76.74
0.50	5	3.41	241.42	1.47	109.23	0.1947	0.4520	3.23	142.51	74.83
0.50	10	3.41	482.84	1.04	154.70	0.0217	0.2840	3.06	249.49	71.11
0.75	2	3.12	104.85	2.17	74.52	0.7569	0.7751	3.08	78.87	85.63
0.75	3	3.12	157.28	1.76	91.40	0.3583	0.5572	3.04	104.69	79.82
0.75	4	3.12	209.71	1.52	105.62	0.1727	0.4440	3.01	129.95	76.63
0.75	5	3.12	262.13	1.36	118.13	0.0760	0.3716	2.98	154.72	74.59
0.75	10	3.12	524.26	0.96	167.20	0.0007	0.1922	2.89	277.16	69.21

Table 2 shows that as expected, the firms bid lower prices as N increases and the performance of the competitive solution deteriorates. Note that for the case of full information, the performance of the competitive game given in (3) gives 85.36, 78.87, 75.00, 72.36 and 65.81 for N=2, 3, 4, 5 and 10, respectively. The performance of the competitive solution with asymmetric information is consistently better than these values and the gap increases as N grows. We also see that the competitive solution performs the best when $q = 0.50$, i.e., when the demand rate variability is highest. An interesting observation is that a high type firm may bid lower than a low type firm (for $N = 2,\ q = 0.25, 0.50$) in equilibrium.

Table 3 **Asymmetric Information:** $\lambda_H = 12,\ \lambda_L = 10,\ K = 80,\ h = 1$

		Independent		Cooperative		Competitive				
q	N	ET^i	ETC^i	ET^c	ETC^c	p_L^g	p_H^g	ET^g	ETC^g	% Perf
0.25	2	3.91	81.91	2.76	57.94	0.9786	0.9742	3.91	61.45	85.37
0.25	3	3.91	122.86	2.26	70.97	0.6382	0.6902	3.91	81.91	78.92
0.25	4	3.91	163.82	1.95	81.96	0.4680	0.5481	3.90	102.35	75.10
0.25	5	3.91	204.77	1.75	91.64	0.3659	0.4627	3.90	122.75	72.50
0.25	10	3.91	409.55	1.23	129.60	0.1621	0.2912	3.89	224.32	66.17
0.50	2	3.83	83.82	2.70	59.30	0.9558	0.9551	3.82	62.89	85.37
0.50	3	3.83	125.73	2.20	72.64	0.6079	0.6647	3.82	83.82	78.94
0.50	4	3.83	167.64	1.91	83.88	0.4341	0.5194	3.81	104.71	75.14
0.50	5	3.83	209.55	1.71	93.79	0.3299	0.4321	3.81	125.56	72.56
0.50	10	3.83	419.09	1.21	132.65	0.1224	0.2566	3.79	229.16	66.31
0.75	2	3.74	85.73	2.64	60.64	0.9313	0.9348	3.74	64.31	85.37
0.75	3	3.74	128.59	2.16	74.28	0.5753	0.6377	3.73	85.72	78.93
0.75	4	3.74	171.45	1.87	85.78	0.3975	0.4891	3.73	107.10	75.11
0.75	5	3.74	214.32	1.67	95.90	0.2910	0.3999	3.73	128.45	72.52
0.75	10	3.74	428.63	1.18	135.64	0.0790	0.2211	3.71	234.65	66.21

We repeat the analysis for $\lambda_L = 10$ and $\lambda_H = 12$ in Table 3. The results are similar except that the performance of the competitive solution is slightly worse. This is again due to the impact of variability.

4. Conclusion

We study a competitive joint replenishment game with N identical firms. In this game, each firm places a bid that specifies how much it is willing to pay per unit of its replenishment and a common supplier determines the joint order cycle length based on these bids. It is shown that, in equilibrium, the firms' bids are such that joint order frequency is same as the order frequencies when the firms order independently. Thus, inventory holding costs remain the same, but the order costs reduce to $1/N$ of the order costs when ordering independently, leading to a total cost of $(N+1)/2N$ of total costs when ordering independently. This is certainly more than $1/\sqrt{N}$ of total costs that can be obtained under cooperation. We extend the model for the case of private information about demand rates and characterize the Bayesian Nash equilibrium. It is shown that information asymmetry may offer some improvements to the performance of the competitive solution.

The model in this paper is fairly simple and can be extended in several important directions. First, the bidding process can be enriched perhaps by allowing each firm to bid a menu of contracts that includes pairs of order frequencies and how much it wants to pay per unit of replenishment for that frequency. Second, capacities may be introduced on replenishment sizes that can be set by the RSP, reflecting, for example, the truck sizes for the transportation and maximum batch sizes in production. Finally, the impact of non–identical firms and minor setup costs can be investigated.

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