Optimal Bundle Formation and Pricing of Two Products with Limited Stock

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We consider the stochastic modeling of a retail firm that sells two types of perishable products in a single period not only as independent items but also as a bundle. Our emphasis is on understanding the bundling practices on the inventory and pricing decisions of the firm. One of the issues we address is to decide on the number of bundles to be formed from the initial product inventory levels and the price of the bundle to maximize the expected profit. Product demands follow a Poisson Process with a price dependent rate. Customer reservation prices are assumed to have a joint distribution. We study the impact of reservation price distributions, initial inventory levels, product prices, demand arrival rates and cost of bundling. We observe that the expected profit decreases as the correlation between the reservation prices of two products increases. With negative correlation, bundling cost has a significant impact on the number of bundles formed. When the product prices are low, the retailer sells individual products as well as the bundle (mixed bundling), when they are high, the retailer sells only bundles (pure bundling). The expected profit and the number of bundles offered decrease as the variance of the reservation price distribution increases. For high starting inventory levels, the retailer reduces bundle price and offers more bundles. The number of bundle sales decreases and the number of individual product sales increases when the arrival rate increases since the need for bundling decreases. Impacts of substitutability and complementarity of products are also investigated. The retailer forms more bundles, or charges higher prices for the bundle or both as the products become more complementary and less substitutable.

Keywords: Revenue management, pricing, product bundling, marketing

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1. Introduction and Literature Review

One of the main objectives of a firm is to maximize its profits. Profits can be increased through decreasing costs or increasing revenues or both. While most of the operational activities of a firm and academic literature in operations research focus on reduction of costs, significant profit improvements are also possible by managing demand and increasing revenue. Recently, several industries and many researchers are turning their attention to the demand side of the equation under the concept of revenue management.

Revenue management deals with the sales of perishable products by controlling price and inventory with the objective of increasing revenue. Some examples of perishable products for which revenue management approaches can provide significant benefits include airline seats, hotel rooms, seasonal and fashion goods, high technology goods, seats for the theater and sporting events, and traffic on network lines. Revenue management has become one of the most successful application areas of operations research and academic research has grown rapidly in recent years (Talluri and van Ryzin 2004).

In this paper, we study product bundling in revenue management. Specifically, we consider the bundle formation and pricing decisions of a retailer that sells a limited stock of two products facing random demand over a finite selling horizon. Bundling is defined as the practice of selling two or more products together. Companies practice bundling in a broad range of industries including information goods (e.g., software such as Microsoft’s Office Suite), travel services (e.g., vacation packages from travel agencies), restaurants (e.g., McDonald’s Happy Meal), durable consumer goods (e.g., personal computer options), and non durable consumer goods (e.g., dishwasher detergent and rinse aid packages). While we are only focusing on pricing efficiencies obtained through bundling, bundles are offered for a variety of other reasons. Strategically, a company may use bundling to preserve (or increase) market power or to extend its market power in one product to another. Efficiency reasons include achieving cost savings and quality improvements. See Nalebuff (2003) for a detailed discussion of the motivations to engage in bundling.

The implementation of bundling may require a number of important and rather difficult decisions. First, the benefits of bundling need to be quantified in order to see whether these benefits justify the potential costs and additional complexity in operations. Also, if the company is offering more than two products, it needs to specify the number of different bundle types to offer and what products to include in each specific bundle. For products that are sold as part of a bundle, the company also needs to decide whether it will continue to sell these products individually (i.e., mixed bundling) or not (i.e., pure bundling). Finally, the company needs to determine the bundle prices and individual product prices that will maximize its profits.

Bundling is categorized as product bundling and price bundling. Stremersch and Tellis (2002) refer to price bundling as “the sale of two or more separate products in a package at a discount, without any integration of products” and to product bundling as “the integration and sale of two or more separate products or services at any price”. Since usually physical
integration needs to take place before the demand for product bundles, bundle formation decisions, i.e., how many units of individual products should be converted to bundles, are also crucial for product bundling.

**Literature**

Previous research on bundling in marketing and economics literature attempts to identify demand settings for which bundling is profitable. The purchase behavior of the customers is usually characterized by the reservation price (maximum price a customer is willing to pay for a product) distributions of the products. Correlation between the reservation prices, complementarity, substitutability, and heterogeneity of valuations among customers are major factors studied in this literature. The earliest study to address such issues is by Stigler (1963) who assumes additive reservation prices for the bundle and concludes that the profitability of bundling is due to the negative correlation in reservation prices. Adams and Yellen (1976) use the same settings and argue that the profitability of bundling can stem from its ability to sort customers into groups with different reservation price characteristics, and hence, extract consumer surplus. Considering the three bundling strategies, unbundling, pure bundling and mixed bundling, they conclude that relative profitability of these three strategies depend on the distribution of the reservation prices and the structure of the costs (see also Jeuland (1984)). In numerous experiments they have provided, it is found that some form of bundling is more profitable than simple monopoly pricing and bundling seems to be a more efficient method than price discrimination. Schmalensee (1984) modifies the framework of Stigler (1963) by assuming bivariate normal reservation price distribution and allowing for positive correlation. He shows that pure bundling operates by reducing the effective dispersion in buyers’ tastes, since the standard deviation of reservation prices for the bundle is less than the sum of the standard deviations for the two components as long as reservation prices are not perfectly correlated. Schmalensee (1984) also shows that mixed bundling combines the advantages of pure bundling and unbundling strategies. This policy enables the seller to reduce effective heterogeneity among those buyers with high reservation prices for both goods, while still selling at a high markup to those buyers willing to pay a high price for only one of the goods. In a comment to Schmalensee (1984), Long (1984) relaxes the normality assumption on reservation price distributions and also concludes that the most favorable case for bundling as a price discrimination device is when the bundle components have negatively correlated reservation prices. Focusing on graphical analysis of bundling, Salinger (1995) indicates that if bundling does not lower costs, it tends to be profitable with negatively correlated reservation prices that are high relative to costs. If bundling lowers costs and costs are high relative to reservation values, positively correlated reservation values increase the incentive to bundle.

Although not directly related to our study, see also Ansari et al. (1996) for the determination of the optimal number of items to be included in a service bundle, Ben-Akiva and Gershenfeld (1998) for customer choice behavior for bundles with correlated demand, Carbajo
et al. (1990) for incentives for bundling under imperfect competition, Hanson and Martin (1990) for the calculation of optimal bundle prices in a deterministic setting, and Stremersch and Tellis (2002) for a clear discussion of bundling terms which are used in marketing, economics, and law literature in a somewhat unclear way. Finally, we note the growing literature on bundling of information goods (see for example, Bakos and Brynjolfsson (1999)). However, the setting for the information goods is distinctly different from physical goods and most services, since the marginal costs are close to zero and inventory is almost never a constraint.

The basic assumption in the related marketing and economics literature is that there is an abundant supply of the products, perhaps at a certain cost. Since supply is not a constraint, there is also no distinction between price bundling and product bundling (except perhaps the reservation price of a product bundle is greater than the sum of individual product reservation prices). Different from them, we assume that there is an initial inventory of items that is to be sold over a finite horizon. Therefore our approach is in line with the approach taken in the revenue management literature. See Talluri and van Ryzin (2004) for a detailed review of revenue management research and Elmaghraby and Keskinocak (2003) for a review of dynamic pricing research and practice in this context.

Inventory considerations in bundling decisions are critical in many product categories including travel services (airplane seats, hotel rooms, and rental cars), event tickets, fashionable products such as apparel and accessories, and high technology products. Since inventory is explicitly a constraint, the distinction between the price and product bundles is also important. In this paper we focus on product bundling and thus, the portion of the initial inventory that will be used to form bundles is also a decision in our model. We also incorporate bundle formation costs in our model. We will also explicitly consider bundle formation cost, random demands, and consumer behavior when the inventory of one of the individual products or the bundle runs out before the end of the horizon.

The papers that could be considered most directly related to our work in the revenue management literature are those studying multiple product revenue management problems as introduced in Gallego and van Ryzin (1997). Netessine et al. (2004) study a problem where they consider an e-commerce seller that dynamically forms and prices product or service packages. The problem is modeled as a dynamic program based on two possibilities in case of stock-out: an emergency replenishment of the customer’s initial request or lost sales. Our model differs from Netessine et al. (2004) as we assume posted prices and product bundles and we explicitly model the consumer choice given that she is given three alternatives upfront: either one of the products or the bundle or none.

Ernst and Kouvelis (1999) study a newsboy type modeling framework where the retail firm sell products not only as independent items but also as part of a bundle (or a packaged good). They study a problem similar to ours, however they focus on the inventory decisions only, taking the price set as a given parameter. They also examine the switching between the individual products and the packaged good when there are stock-outs. Through a numerical study, they show that positive correlation of original demands favors increased stocking levels
of a multi-product package, while negative correlation tends to have the opposite effect. They also show that correlated demands result in higher profitability and stronger substitutability results in higher stocking levels for the packaged good.

Bulut et al. (2005) study the single period pricing of two perishable products which are sold individually and as a bundle. Their customer demand has a Poisson distribution with a price dependent arrival rate. Assuming a general reservation price distribution, they determine the optimal product prices that maximize the expected revenue. They also compare the performance of different bundling strategies under different conditions such as different reservation price distributions, demand arrival rates, and starting inventory levels. Their numerical study demonstrates that, when individual product prices are fixed to high values, the expected revenue is a decreasing function of the correlation coefficient, while for low prices the expected revenue is an increasing function of the correlation coefficient. They indicate that, bundling is least effective in case of limited supply and the mixed bundling strategy outperforms the others, especially when the customer reservation prices are negatively correlated. The major distinction between this work and ours is that we assume product bundles (and thus bundle formation decisions are also of interest) and we allow for customer switches to take place at the end of the horizon.

Scope of our study and summary of results

We consider a retail firm that sells two types of perishable products over a finite selling season. The starting inventory levels of these two products are fixed and at the beginning of the season, the retailer uses all or a portion of these initial stocks to form product bundles. The retailer then sells the bundles as well as the individual products (mixed bundling) by charging constant prices over the selling season. We determine the optimal number of product bundles that the retailer should form and the optimal individual and bundle prices that the retailer should charge so as to maximize his expected profit over the selling season. No replenishments are allowed during the selling season, and separation of bundles into individual items is not allowed. We also investigate the effect of cost of forming bundles, as these costs could be non negligible in some industries. For example, combining separate PC components into a PC requires technicians to work on, which adds a labor cost to bundling (See Ansari et al. (1996) for a model to determine the number of bundles to be formed).

Customer arrivals follow a Poisson process with a constant arrival rate and their choice of individual products or the product bundle is governed by their reservation prices. “Posted” product prices are used which means prices are known by the customers, however they do not know the available inventory before they actually arrive at the store. When they arrive, if their preferred product is not available, they may switch to another product or do not make a purchase. These switching probabilities depend on the reservation prices and posted prices set by the retailer.

Using a numerical study, we investigate the effects of various factors on our model, such as the correlation between the reservation price distributions, the variance of the reservation price distributions, initial inventory levels, the bundle formation cost, and the intensity
of the customer arrivals. We observe that the expected profit decreases as the correlation coefficient increases and increase in bundling cost lowers the profit. With negative correlation, bundling cost has a significant impact on the number of bundles formed. However with positive correlation, this effect is negligible.

When the product prices are below the mean value of the reservation price distribution, the retailer sets a high bundle price and sells individual products as well as the bundle. When the individual products are high, the retailer charges a bundle price such that only bundles are sold. The expected profit and the optimal number of bundles formed decrease as the variance of the reservation price distribution increases. However, the optimal bundle price has different a behavior depending on the correlation of the reservation prices. For negative correlation, optimal bundle price decreases as the standard deviation increases. For positive correlation, the optimal bundle price is an increasing function of the standard deviation. For the uncorrelated case, the optimal bundle price is a decreasing function of the standard deviation for small bundling cost values and it is an increasing function for large bundling cost values.

Finally we perform numerical analysis to investigate the impact of product substitutability and complementarity. For all correlation values and bundle formation costs, we see that the retailer is forming more bundles, or charging higher prices for the bundle, or both as the products become more complementary and less substitutable. The impact of substitutability or complementarity is more pronounced when the product prices are uncorrelated and positively correlated.

The rest of the paper is organized as follows. Section 2 formulates the problem investigated and explains the stochastic model used in our study. In Section 3 we give results of our numerical studies and in the final section we conclude with the discussion of our major findings and the avenues for future research.

2. Model and the Analysis

We consider a retailer that sells two product types (product 1 and product 2) and a bundle (which is formed with the two products) over a finite selling season. Initial inventory levels, \( Q_1 \) and \( Q_2 \), of products 1 and 2 are given and the retailer decides the number of bundles (\( n_b \)) to be formed with this initial inventory with a unit bundling cost of \( c \). Once the number of bundles to be formed at the beginning of the season is decided, no new bundles are formed and none of them are unbundled to offer individual products during the season.

A customer is allowed to buy only one type of product, which means he or she can buy one unit of product 1, product 2 or a bundle but not any combination of the products. The customer can also leave the store without buying any product. The prices of product 1, product 2 and the bundle (\( p_1 \), \( p_2 \) and \( p_b \)) are determined such that \( p_b \leq p_1 + p_2 \). Prices are set at the beginning of the selling season and they are fixed during the period. The objective of the retailer is to maximize the expected profit over this single selling season.

Customer arrivals to the store follow a Poisson process with a fixed arrival rate of \( \lambda \) per season. The decision to buy a product is determined by the comparison of the customer’s
reservation price with the product prices. Purchase probabilities for products 1, 2 and the
bundle are denoted as \( m_1(p_1, p_2, p_b) \), \( m_2(p_1, p_2, p_b) \) and \( m_b(p_1, p_2, p_b) \), respectively. For brevity,
we express \( m_i(p_1, p_2, p_b) \) as \( m_i \) for \( i = 1, 2, b \). Then \( m_0 = 1 - m_1 - m_2 - m_b \) is the probability
of no purchase. The arrival rates \( \ell_1, \ell_2 \) for the two products and the bundle \( \ell_b \) are then given
as \( \ell_i = \lambda * m_i \) for \( i = 1, 2, b \).

Customer reservation prices are random variables. Reservation price of product 1 is \( R_1 \)
and reservation price of product 2 is \( R_2 \). Mean and variance parameters of the reservation
price distribution are \((\mu_1, \sigma_1)\) and \((\mu_2, \sigma_2)\), for product 1 and product 2, respectively. We now
discuss main assumptions used in our model. First, in this study we are not concerned with
estimating reservation price distributions. We believe that this merits a separate study and
we refer the reader to Jedidi and Zhang (2002) for an example. Our model in the first part
is structured on the main assumption that reservation price for the bundle is the sum of the
reservation prices of individual products that form the bundle, i.e., \( R_b = R_1 + R_2 \). This is
one of the common assumptions used in the bundling literature such as Adams and Yellen
as “the assumption of strict additivity”. In the last part, we relax this assumption following
a model in Venkatesh and Kamakura (2003) and analyze complementary products leading to
superadditive reservation prices and substitutable products leading to subadditive reservation
prices.

Considering the stock-out situations, we assume the following: when one product incurs
shortage, the customer either switches to one of the other products or leaves the store without
any purchase. For these cases we assume that switching behavior will follow a Multinomial
distribution since there are three possible choices. If two types of products incur shortage,
the customer has the alternatives to switch to the available product or to leave without any
purchase; therefore binomial distribution is used to calculate the switching probabilities.

**Purchasing probabilities**

Let \( f_{R_1, R_2}(r_1, r_2) \) denote the joint reservation price density for the two products. When all
products are available, purchasing probabilities are calculated by comparing the customer
reservation price with the product price. A customer buys either a single product or a bundle
or leaves the store without buying any product. Probability expressions of these events are
stated below:

**Probability of no purchase:** A customer will leave the store without buying any product
when his reservation prices for each product and the bundle are all lower than their corre-
sponding sales prices. Therefore, the probability of no purchase is stated as,

\[
m_0 = P(R_1 \leq p_1; R_2 \leq p_2; R_b \leq p_b)
    = P(R_1 \leq p_1; R_2 \leq \min \{p_2; p_b - R_1\})
    = \int_{p_1}^{p_1} \int_{-\infty}^{\min \{p_2; p_b - R_1\}} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2
\]
where $a_1 = \min \{p_2, p_b - r_1\}$.

**Probability of purchasing product 1:** Purchase probabilities are calculated by comparing the consumer surplus, which is the difference between the reservation price and the product price. A customer will purchase product 1 if his surplus from product 1 is positive and greater than his surplus values from other products. Thus the probability of purchasing product 1 is stated as,

$$m_1 = P(R_1 \geq p_1; R_1 - p_1 \geq R_2 - p_2; R_1 - p_1 \geq R_b - p_b)$$

$$= \int_{p_1}^{\infty} \int_{-\infty}^{a_2} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2$$

where $a_2 = \min \{r_1 - p_1 + p_2, p_b - p_1\}$.

**Probability of purchasing product 2:** Similarly,

$$m_2 = P(R_2 \geq p_2; R_2 - p_2 \geq R_1 - p_1; R_2 - p_2 \geq R_b - p_b)$$

$$= \int_{p_2}^{\infty} \int_{-\infty}^{a_3} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1$$

where $a_3 = \min \{r_2 - p_2 + p_1, p_b - p_2\}$.

**Probability of purchasing a bundle:** Again using the same reasoning,

$$m_b = P(R_b \geq p_b; R_b - p_b \geq R_1 - p_1; R_b - p_b \geq R_2 - p_2)$$

$$= \int_{p_b - p_2}^{\infty} \int_{a_4}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2$$

where $a_4 = \max \{p_b - r_1, p_b - p_1\}$

**Switching probabilities**

We now consider situations when one or two products run out of stock during the selling season. We are interested in calculating the probability that a customer who had originally intended to purchase a particular product $i$ will switch to another product $j$, if product $i$ runs out of stock during the selling season. We call these switching probabilities and denote them by $\gamma_{ij}$ if only product $i$ ran out of stock and the customer potentially has two products to choose from or by $\gamma_{ij}^-$ if two products ran out of stock and the customer only has product $j$ to switch to.

**One type of product incurs shortage**

**Probability of switching from product 1 to bundle or to product 2**

Suppose product 1 incurs shortage and the customer has the option to switch to the bundle, product 2 or to leave without purchase. Let $\gamma_{1B}$, $\gamma_{12}$ and $1 - \gamma_{1B} - \gamma_{12}$ be the probability of these events respectively. Since the switching events follow after the first choices of the customers are made, we need to calculate these probabilities conditional on the event that the first preference of the customer was to buy product 1. Then we have

$$\gamma_{1B} = \Pr \{R_b - p_b \geq R_2 - p_2; R_b \geq p_b | R_1 - p_1 \geq R_b - p_b; R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \}$$
= \Pr \{ R_1 \geq \max (p_b - p_2, p_1); \min (p_b - p_1, R_1 - p_1 + p_2) \geq R_2 \geq p_b - R_1 \} / m_1

= \Pr \{ R_1 \geq \max (p_b - p_2, p_1); p_b - p_1 \geq R_2 \geq p_b - R_1 \} / m_1

= \int_{\max(p_b-p_2,p_1)}^{\infty} \int_{p_b-r_1}^{p_b} f_{R_1,R_2}(r_1,r_2)dr_1dr_2/m_1

Similarly we have

\gamma_{12} = \Pr \{ R_2 - p_2 \geq R_b - p_b; R_2 \geq p_b \mid R_1 - p_1 \geq R_b - p_b; R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \}

= \int_{p_b-p_2}^{\infty} \int_{p_2+p_1}^{p_2} f_{R_1,R_2}(r_1,r_2)dr_1dr_2/m_1

**Probability of switching from product 2 to bundle or to product 1**

Similar to the previous case let \( \gamma_{2B} \), \( \gamma_{21} \) be the probability of switching to a bundle or to product 1 when product 2 stocks out. We have

\gamma_{2B} = \Pr \{ R_b - p_b \geq R_1 - p_1; R_b \geq p_b \mid R_2 - p_2 \geq R_b - p_b; R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \}

= \int_{p_b-p_2}^{\infty} \int_{p_2+p_1}^{\infty} f_{R_1,R_2}(r_1,r_2)dr_1dr_2/m_1

and

\gamma_{21} = \Pr \{ R_1 - p_1 \geq R_b - p_b; R_1 \geq p_1 \mid R_2 - p_2 \geq R_b - p_b; R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \}

= \int_{p_2}^{\infty} \int_{p_1}^{p_2+p_1} f_{R_1,R_2}(r_1,r_2)dr_1dr_2/m_2

**Probability of switching from bundle to product 1 or to product 2**

Now let \( \gamma_{B1} \) and \( \gamma_{B2} \) be the probability of switching to product 1 or to product 2 when bundle stocks out. We have

\gamma_{B1} = \Pr \{ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \mid R_b - p_b \geq R_1 - p_1; R_b - p_b \geq R_2 - p_2; R_b \geq p_b \}

= \int_{\max(p_1,p_b-p_2)}^{\infty} \int_{p_b-p_1}^{r_1+p_2} f_{R_1,R_2}(r_1,r_2)dr_1dr_2/m_b

and

\gamma_{B2} = \Pr \{ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \mid R_b - p_b \geq R_1 - p_1; R_b - p_b \geq R_2 - p_2; R_b \geq p_b \}

= \int_{\max(p_2,p_b-p_1)}^{\infty} \int_{p_b-p_2}^{r_2+p_1} f_{R_1,R_2}(r_1,r_2)dr_1dr_2/m_b

**Two types of products incur shortage**

**Product 1 and product 2 incur shortage:**

Let \( \gamma_{1B} \), \( \gamma_{2B} \) be the probability of switching from product 1 or product 2 to bundle when only bundle is available. Then we have,

\gamma_{1B} = \Pr \{ R_b \geq p_b \mid R_1 - p_1 \geq R_b - p_b; R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \}

= \Pr \{ R_1 + R_2 \geq p_b; R_1 - p_1 \geq R_1 + R_2 - p_b; R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \} / m_1

= \Pr \{ R_1 \geq p_1; \min(p_b - p_1, R_1 - p_1 + p_2) \geq R_2 \geq p_b - R_1 \} / m_1

= \int_{p_1}^{\infty} \int_{\min(p_b-p_1,R_1-p_1+p_2)}^{\infty} f_{R_1,R_2}(r_1,r_2)dr_1dr_2/m_1
and
\[
\gamma_{2b} = \Pr \{ R_b \geq p_b \mid R_2 - p_2 \geq R_b - p_b; R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \} = \int_{p_2}^{\infty} \int_{p_b - r_2}^{\min(r_2 - p_2 + p_b - p_2)} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_2
\]

**Bundle and product 2 incur shortage**

Let \( \gamma_{21} = \gamma_{21} \) be the probability of switching from bundle or product 2 to product 1 when only product 1 is available. Then
\[
\gamma_{21} = \Pr \{ R_1 \geq p_1 \mid R_b - p_b \geq R_2 - p_2; R_b - p_b \geq R_1 - p_1; R_2 \geq p_2 \} = \int_{\max(p_1, p_b - p_2)}^{\infty} \int_{p_b - p_2}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_b
\]
and we have
\[
\gamma_{21} = \Pr \{ R_1 \geq p_1 \mid R_2 - p_2 \geq R_b - p_b; R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \} = \int_{p_1}^{\infty} \int_{r_1 - p_1 + p_2}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_2
\]

**Bundle and product 1 incur shortage**

Similar to the previous case let \( \gamma_{12} = \gamma_{12} \) be the probability of switching when only product 2 is available. Then,
\[
\gamma_{12} = \Pr \{ R_2 \geq p_2 \mid R_b - p_b \geq R_1 - p_1; R_b - p_b \geq R_2 - p_2; R_b \geq p_b \} = \int_{p_b - p_2}^{\infty} \int_{\max(p_2, p_b - p_1)}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_b
\]
and
\[
\gamma_{12} = \Pr \{ R_2 \geq p_2 \mid R_1 - p_1 \geq R_b - p_b; R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \} = \int_{p_1}^{\infty} \int_{r_2 - p_2 + p_1}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_2
\]

**Sales probabilities and the objective function**

Recall that \( Q_1 \) and \( Q_2 \) are the initial inventory levels of product 1 and product 2, respectively. Let \( n_b \) be the number of bundles formed and \( n_i \quad i = 1, 2 \) be the remaining units of product \( i \), with \( n_1 = Q_1 - n_b \) and \( n_2 = Q_2 - n_b \).

Also let \( X_1 \), \( X_2 \), and \( X_b \) denote the number of customers whose first preference are for product 1, product 2, and the bundle respectively. We also have the random variables corresponding to the number of customers that switch from one product to another. These variables are denoted as \( X_{ij} \), where \( i \) is the customer initial preference and \( j \) is the type of the product that the customer switches to (substitutes for \( i \)).

The realized values of these random variables will be denoted by \( x_1 \), \( x_2 \), \( x_b \), \( x_{1b} \), \( x_{12} \), \( x_{2b} \), \( x_{21} \), \( x_{b1} \) and \( x_{b2} \). Let
\[
P \left( x_1, x_2, x_b, x_{1b}, x_{12}, x_{2b}, x_{21}, x_{b1}, x_{b2} \right) = P \left( X_1 = x_1, X_2 = x_2, X_b = x_b, X_{1b} = x_{1b}, X_{12} = x_{12}, X_{2b} = x_{2b}, X_{21} = x_{21}, X_{b1} = x_{b1}, X_{b2} = x_{b2} \right)
\]
Case 1: No shortage occurs
Case 2: Bundle incurs shortage
   a) All excess demand of the bundle is satisfied
   b) Product 1 incurs shortage with the excess bundle demand
   c) Product 2 incurs shortage with the excess bundle demand
   d) Both products incur shortage with the excess bundle demand
Case 3: Product 1 incurs shortage
   a) All excess demand of product 1 is satisfied
   b) Product 2 incurs shortage with the excess demand of product 1
   c) Bundle incurs shortage with the excess demand of product 1
   d) Product 2 and bundle incur shortage with the excess demand of product 1
Case 4: Product 2 incurs shortage
   a) All excess demand of product 2 is satisfied
   b) Product 1 incurs shortage with the excess demand of product 2
   c) Bundle incurs shortage with the excess demand of product 2
   d) Product 1 and bundle incur shortage with the excess demand of product 2
Case 5: Product 1 and the bundle incur shortage
   a) All excess demand of product 1 and the bundle are satisfied
   b) Product 2 incurs shortage with the excess demand of product 1 and the bundle
Case 6: Product 2 and the bundle incur shortage
   a) All excess demand of product 2 and the bundle are satisfied
   b) Product 1 incurs shortage with the excess demand of product 2 and the bundle
Case 7: Product 1 and product 2 incur shortage
   a) All excess demand of the products are satisfied from the bundle
   b) Bundle incurs shortage with the excess demand of two products
Case 8: All products incur shortage

Table 1   Cases of realizations to derive the expected profit

denote the joint probability mass function of those random variables. Note that for certain realizations we only need joint marginal probability mass function of only a subset of these variables.

When all dedicated demand can be satisfied, due to the independence property of the Poisson processes, we have

\[ P(X_1 = x_1, X_2 = x_2, X_b = x_b) = P(X_1 = x_1)P(X_2 = x_2)P(X_b = x_b) \]

where \( X_i \) has a Poisson distribution with rate \( \ell_i = \lambda * m_i, \ i = 1, 2, b \).

For the derivation of the expected profit, \( \pi \), there are eight possible realizations when only the initial choices are considered. After this first classification, sub-cases are defined to include the switching customer realizations. All realization cases are listed in Table 1.

We have two assumptions regarding the allocation of inventory and customer switches to simplify the analysis. First, we assume that the inventory of a product (or the bundle) is first allocated to customers whose first choice is that product (or the bundle). The customers that cannot find their first choices switch to their second choice (if their surplus is also positive for the second choice) and the demands resulting from these switches are satisfied with the remaining inventory of their second choice. However, if the second choice also runs out of
stock, we assume that there are no further switches. We now will provide the expressions of these realizations. Due to space limitations we only provide the details of Case 1, Case 2, Case 5 and Case 8. Cases 3 and 4 can be derived similar to Case 2. Cases 6 and 7 can be derived similar to Case 8.

**CASE 1: NO SHORTAGE OCCURS \((x_1 \leq n_1, x_2 \leq n_2, x_b \leq n_b)\)**

Expected profit in the region where all customers are satisfied by their first choice products is given by \(\pi_1\) which is

\[
\pi_1 = \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} (p_1 x_1 + p_2 x_2 + p_b x_b - c n_b) P(x_1, x_2, x_b)
\]

\[
= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} (p_1 x_1 + p_2 x_2 + p_b x_b - c n_b) \frac{e^{-\ell f_1} e^{-\ell f_2} e^{-\ell f_b}}{x_1! x_2! x_b!}.
\]

**CASE 2: BUNDLE INCURS SHORTAGE \((x_1 \leq n_1, x_2 \leq n_2, x_b > n_b)\)**

Initial demand for the bundle is more than the available stock but initial demands for product 1 and product 2 are satisfied from the stocks. Excess demand of the bundle customers can result in four sub-cases.

(a) **All excess demand of the bundle is satisfied**

Let \(x_b - n_b\) be the number of excess bundle customers. In this case, \(x_b - n_b\) units are satisfied from the excess inventories of product 1 and product 2. That is, we have \(x_1 + x_{b_1} \leq n_1\), \(x_2 + x_{b_2} \leq n_2\), \(x_1 \leq n_1, x_2 \leq n_2, x_b > n_b\) and the contribution of this case to the total expected profit is given by:

\[
\pi_{2a} = \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_{b_1}=0}^{x_b-n_b} \sum_{x_{b_2}=0}^{x_b-n_b} (p_1 (x_1 + x_{b_1}) + p_2 (x_2 + x_{b_2}) + p_b n_b - c n_b) \frac{e^{-\ell f_1} e^{-\ell f_2} e^{-\ell f_b}}{x_1! x_2! x_b!} (x_{b_1}! x_{b_2}! (\gamma B_1)^{x_{b_1}} (\gamma B_2)^{x_{b_2}} (\gamma B_0)^{x_{b_0}})
\]

where \(x_{b_0} = x_b - n_b - x_{b_1} - x_{b_2}\) and \(\gamma B_0 = 1 - \gamma B_1 - \gamma B_2\).

(b) **Product 1 incurs shortage with the excess bundle demand**

In this case, original demand for product 1 is satisfied but the left over is not sufficient to satisfy the overflow from the bundle customers. All demands for product 2 are satisfied from the stock. For this case we have \(x_1 + x_{b_1} > n_1\), \(x_2 + x_{b_2} \leq n_2\), \(x_1 \leq n_1, x_2 \leq n_2, x_b > n_b\) and the contribution is given by:

\[
\pi_{2b} = \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_{b_1}=0}^{\infty} \sum_{x_{b_2}=0}^{x_b-n_b} (p_1 n_1 + p_2 (x_2 + x_{b_2}) + p_b n_b - c n_b) \frac{e^{-\ell f_1} e^{-\ell f_2} e^{-\ell f_b}}{x_1! x_2! x_b!} (x_{b_1}! x_{b_2}! x_{b_0}! (\gamma B_1)^{x_{b_1}} (\gamma B_2)^{x_{b_2}} (\gamma B_0)^{x_{b_0}})
\]
(c) **Product 2 incurs shortage with the excess bundle demand**

This case is similar to the above case, except that product 2 incurs shortage. Hence we have \( x_1 + x_{b1} \leq n_1, \ x_2 + x_{b2} > n_2, \ x_1 \leq n_1, \ x_2 \leq n_2, \ x_b > n_b \) and the contribution is given by:

\[
\pi_{2c} = \sum_{n_1} \sum_{n_2} \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_{b1}=0}^{x_1-n_b} \sum_{x_{b2}=0}^{x_2-n_b} \left( p_1(x_1 + x_{b1}) + p_2n_2 + p_bn_b - c_n_b \right)
\]

\[
e^{-\ell_1x_1} \frac{x_1!}{x_1!} e^{-\ell_2x_2} \frac{x_2!}{x_2!} e^{-\ell_bx_b} \frac{x_b!}{x_b!} (x_b - n_b)! \left( \beta_{B1}x_{b1} \beta_{B2}x_{b2} \beta_{B0}x_{b0} \right)
\]

(d) **Both products incur shortage with the excess bundle demand**

In this case the excess inventories of product 1 and product 2 are not sufficient to satisfy the overflow demand from the bundle customers. That is, we have \( x_1 + x_{b1} > n_1, \ x_2 + x_{b2} > n_2, \ x_1 \leq n_1, \ x_2 \leq n_2, \ x_b > n_b \) and

\[
\pi_{2d} = \sum_{n_1} \sum_{n_2} \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_{b1}=0}^{x_1-n_b} \sum_{x_{b2}=0}^{x_2-n_b} \left( p_1n_1 + p_2n_2 + p_bn_b - c_n_b \right)
\]

\[
e^{-\ell_1x_1} \frac{x_1!}{x_1!} e^{-\ell_2x_2} \frac{x_2!}{x_2!} e^{-\ell_bx_b} \frac{x_b!}{x_b!} (x_b - n_b)! \left( \beta_{B1}x_{b1} \beta_{B2}x_{b2} \beta_{B0}x_{b0} \right)
\]

Expected profit for the Case 2 is calculated as \( \pi_2 = \pi_{2a} + \pi_{2b} + \pi_{2c} + \pi_{2d} \).

**CASE 5: PRODUCT 1 AND BUNDLE INCUR SHORTAGE (x_1 > n_1, x_2 \leq n_2, x_b > n_b)**

Initial demands for product 1 and the bundle are greater than the respective stock amounts and only initial demand for product 2 is satisfied from the stock. Excess demands of product 1 and the bundle result in two sub-cases.

(a) **All excess demand of product 1 and the bundle are satisfied**

Demand for product 2 is satisfied including the switching customers from product 1 and the bundle. That is, we have \( x_2 + x_{12} + x_{b2} \leq n_2, \ x_1 > n_1, \ x_2 \leq n_2, \ x_b > n_b \) and

\[
\pi_{5a} = \sum_{x_1=n_1+1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_{b1}=0}^{x_1-n_b} \sum_{x_{b2}=0}^{x_2-n_b} \left( p_1n_1 + p_2(x_2 + x_{12} + x_{b2}) + p_bn_b - c_n_b \right)
\]

\[
e^{-\ell_1x_1} \frac{x_1!}{x_1!} e^{-\ell_2x_{12}} \frac{x_2!}{x_2!} e^{-\ell_bx_b} \frac{x_b!}{x_b!} (x_b - n_b)! \left( \gamma_{12} \gamma_B x_{b1} x_{b2} (1 - \gamma_{12}) x_{12} \right)
\]

(b) **Product 2 incurs shortage with the excess demand of product 1 and bundle**

Initial product 2 demand is satisfied but excess demand from product 1 and bundle cannot be satisfied with the product 2 stock. We have \( x_2 + x_{12} + x_{b2} > n_2, \ x_1 > n_1, \ x_2 \leq n_2, \ x_b > n_b \) and

\[
\pi_{5b} = \sum_{x_1=n_1+1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_{b1}=0}^{x_1-n_b} \sum_{x_{b2}=0}^{x_2-n_b} \left( p_1n_1 + p_2n_2 + p_bn_b - c_n_b \right)
\]

\[
e^{-\ell_1x_1} \frac{x_1!}{x_1!} e^{-\ell_2x_{12}} \frac{x_2!}{x_2!} e^{-\ell_bx_b} \frac{x_b!}{x_b!} (x_b - n_b)! \left( \gamma_{B2} x_{b1} (1 - \gamma_{B2}) x_{12} \right)
\]
\[
\frac{e^{-\ell_1}x_1!}{x_1!} e^{-\ell_2}x_2! e^{-\ell_b}x_b! \left( x_1 - n_1 + x_{12} - 1 \right) \left( x_2 - n_2 + x_{b2} - 1 \right) \left( x_b - n_b + x_{b2} - 1 \right) \left( 1 - \gamma_{12} \right)^{x_1-n_1} \left( 1 - \gamma_{2b} \right)^{x_b-n_b} \left( 1 - \gamma_{12} \right)^{x_{12}} \]

Expected profit for the Case 5 is calculated as \( \pi_5 = \pi_{5a} + \pi_{5b} \).

**CASE 8: ALL PRODUCTS INCUR SHORTAGE**

As the last case, we consider the case where all products incur shortage with the initial dedicated customer demand. That is \( x_1 > n_1, x_2 > n_2, x_b > n_b \) and

\[
\pi_8 = \sum_{x_1=n_1+1}^{\infty} \sum_{x_2=n_2+1}^{\infty} \sum_{x_b=n_b+1}^{\infty} (p_1n_1 + p_2n_2 + p_bn_b) e^{-\ell_1}x_1! e^{-\ell_2}x_2! e^{-\ell_b}x_b! \frac{p_1n_1 + p_2n_2 + p_bn_b}{x_1! x_2! x_b!}.
\]

The total expected profit is calculated as adding all expected profits calculated for each cases. It can be formulated as:

\[
\pi = \sum_{i=1}^{8} \pi_i
\]

where \( i \) is the index for the cases.

**Superadditivity and subadditivity of reservation prices**

The analysis so far assumes the strict additivity of customer reservation prices where reservation price for the bundle is the sum of the reservation prices of individual products that form the bundle. When the products are complements or substitutes this assumption no longer holds. In order to model substitutability or complementarity, we use the model in Venkatesh and Kamakura (2003) who refer to the degree of substitutability and complementarity as the degree of contingency, \( \theta \), and define it as:

\[
\theta = \frac{R_b - (R_1 + R_2)}{R_1 + R_2}
\]

When the individual products are complements, reservation price of the bundle is superadditive or \( \theta > 0 \). When the individual products are substitutes, reservation price of the bundle is subadditive, or \( \theta < 0 \). Clearly, \( \theta = 0 \) refers to the strict additive case.

A non–zero degree of contingency changes the purchasing \( (m_i) \) and switching \( (\gamma_{ij}, \gamma_{ij}^-) \) probability expressions since we now have to use \( R_b = (1 + \theta)(R_1 + R_2) \). Purchasing and switching probability expressions for non–zero values of \( \theta \) are provided in Appendix 1. Note that sales probability and objective function formulations that are provided earlier are still valid for non–zero values of \( \theta \).
3. Numerical Study

In this section, we present the results of our numerical study to demonstrate the effects of various factors on the bundling and pricing problem. The factors considered are the correlation between the reservation price distributions, the variance of the reservation price distributions, initial inventory levels, the unit bundle formation cost, and the intensity of the customer arrivals. We also consider the effects of the degree of product complementarity and substitutability, also known as the degree of contingency, in the last part.

For the numerical study, we wrote a FORTRAN code to calculate the expected profit and to determine the optimum number of bundles to be formed and the optimum bundle price for a given parameter set.

Before providing the results, we will discuss the setup of the model used for the numerical study. The model defined in previous section contains a general bivariate continuous distribution for the customer reservation prices. In literature, personal choices and tastes are modeled using distributions of Gaussian family (Schmalensee 1984). Hence, for the numerical study, we choose bivariate normal distribution to model customer reservation prices of the two products. Another reason to use normal distribution to model customer reservation prices of the two products. Another reason to use normal distribution in our model is that sum of two normally distributed random variables is also a random variable with normal distribution. It is also easy to capture correlation effects with bivariate normal distributions. One disadvantage of using normal distribution is that it can take negative values. In our study we avoid this problem by using appropriate parameters that will give nonnegative valuations.

As stated before $R_1$ and $R_2$ denote the reservation prices of product 1 and product 2 and $f_{R_1,R_2}(r_1,r_2)$ is the joint probability density function of these two variables. The joint probability has a bivariate normal distribution defined as,

\[
f_{R_1,R_2}(r_1,r_2) = \frac{e^{-\Gamma(r_1,r_2)/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}},
\]

where $\rho$ is the correlation coefficient between the reservation prices and

\[
\Gamma(r_1,r_2) = \frac{1}{1-\rho^2}[(\frac{r_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{r_1-\mu_1}{\sigma_1})(\frac{r_2-\mu_2}{\sigma_2}) + (\frac{r_1-\mu_1}{\sigma_1})^2].
\]

In this setup, reservation prices of product 1 and product 2 are normally distributed with distribution parameters of $(\mu_1, \sigma_1)$ and $(\mu_2, \sigma_2)$, respectively. Then, for $i = 1, 2$ we have the following marginal distributions for the reservation prices $R_i$, with mean $\mu_i$ and standard deviation $\sigma_i$:

\[
f_{R_i}(r) = \frac{e^{-(r-\mu_i)^2/2\sigma_i^2}}{\sigma_i\sqrt{2\pi}}
\]

Since the problem contains bivariate normality, when $R_b = R_1 + R_2$ (bundle reservation price is strictly additive), $R_b$ is normally distributed with mean $\mu_b = \mu_1 + \mu_2$ and standard deviation $\sigma_b$ calculated as:

\[
\sigma_b = (\sigma_1 + \sigma_2)\sqrt{1 - 2(1-\rho)\omega(1-\omega)}
\]
where \( \omega = \sigma_1 / (\sigma_1 + \sigma_2) \).

In the second part of the study, we consider the substitutability and complementarity of the products. To investigate the effects of factors defined at the beginning of the section, we fix the individual product prices \( p_1 \) and \( p_2 \), and optimize the number of bundles formed, \( n_b \) and the bundle price, \( p_b \). However the model defined in the previous section is a general model and can be used to jointly optimize all product prices.

**The base case**

A base case is used to investigate the effects of the factors. The results of various cases are compared with the base case results. Individual product prices, \( p_1 \) and \( p_2 \) are set to 10 for the base case. Reservation price distributions are such that the marginals have mean value of \( \mu_1 = \mu_2 = 10 \) and standard deviation of \( \sigma_1 = \sigma_2 = 2 \). Degree of contingency, \( \theta \) in this section is zero. Initial inventory level for individual products are \( Q_1 = Q_2 = 5 \). Arrival rate, \( \lambda \) is set to 10. We search optimal bundle price within the search interval with increments of 0.25. We take unit bundling cost \( c \) as 0, 1, 2 and 4. The results of the base case under the correlation coefficient, \( \rho \), of -0.9, 0 and 0.9 are tabulated in Table 2. \( n_b^* \) is the optimal number of bundles and \( p_b^* \) is the optimal bundle price. The fifth column is the expected profit, and the next two columns represent the expected number of products sold. The last three columns are the arrival rate of customers dedicated to each product \( (\ell_1, \ell_2, \ell_b) \) and the arrival rate of customers who would leave the store without buying any product \( (\ell_0) \) even when all products are available.

<table>
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<th>( c )</th>
<th>( n_b^* )</th>
<th>( p_b^* )</th>
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<th>( E(x_b) )</th>
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**Table 2** **Base case**

We also demonstrate the results of the base case in Figure 1. It is observed that the correlation coefficient, \( \rho \), and the bundling cost, \( c \), have significant effects on the optimal number of bundles formed, the bundle price, and the expected profit. The expected profit decreases as the correlation coefficient increases, since the standard deviation of the bundle, \( \sigma_b \) gets larger as the correlation increases. Obviously, increase in bundling cost results in decreasing profits. With negative correlation, bundling cost has a significant impact on the number of bundles formed. However with positive correlation, the solution calls for converting all inventory into the bundle, and only when \( c = 4 \), one unit of each product is spared for individual purchase.
Another finding is that the optimal bundle price is a decreasing function of correlation coefficient.

The impact of individual product prices

To investigate the effects of individual product prices, we set product prices equal to each other \((p_1 = p_2)\) and have values 8, 10 and 12. The values of 8 and 12 are chosen to represent one standard deviation above and below the mean. The other parameters are same as the base case \((\mu_1 = \mu_2 = 10, \sigma_1 = \sigma_2 = 2, Q_1 = Q_2 = 5, \lambda = 10, \rho = -0.9, 0, 0.9, c = 0, 1, 2, 4)\). The results are tabulated in Table 3. We provide three figures extracted from Table 3 (this and the rest of the tables are in Appendix 2). Figure 2 is the profit versus price graph under the correlation coefficient of -0.9 and Figure 3 and Figure 4 are the same graphs under the correlation coefficient of 0 and 0.9 respectively.

When the product prices are below the mean value of the reservation price distribution, the optimal bundle price is the maximum of the possible bundle prices which is the sum of individual product prices \((8+8=16)\). In this case, the retailer sells individual products as well as the bundle (mixed bundling). When the individual products are high, the retailer charges a bundle price such that only bundles are sold (pure bundling). Maximum expected profit for high product prices is very close to the expected profits for medium prices. However expected profit decrease severely, as the bundling cost increases. Similar to our previous results, increase in correlation coefficient results in decrease in the expected profit value for all price levels and the optimal bundle price is a decreasing function of the correlation coefficient.

The impact of initial inventory level

We now study the effect of initial inventory levels on the expected profit. We set up the model with equal individual product inventory levels \(Q_1 = Q_2\) with values 5 and 10. The other parameters are same as the base case \((\mu_1 = \mu_2 = 10, \sigma_1 = \sigma_2 = 2, p_1 = p_2 = 10, \lambda = 10, \rho = -0.9, 0, 0.9, c = 0, 1, 2, 4)\). The results are tabulated in Table 4.
Figure 2  Profit vs. product price for $\rho = -0.9$

Figure 3  Profit vs. product price for $\rho = 0$

Figure 4  Profit vs. product price for $\rho = 0.9$
From this analysis, we see that the expected profit increases as we increase the initial inventory level. This is obvious since we are not considering the cost of purchasing the items. With more inventories, the retailer reduces its bundle price, and also offers more bundles for sale since the customer arrival rate is constant. The results are particularly interesting for $\rho = -0.9$. In this case the retailer converts all inventory to the bundles regardless of the bundle formation cost, as bundling is very effective with negative correlation. For high initial inventory levels, the optimal bundle price decreases more rapidly than the case of low initial inventory levels as the correlation coefficient increases.

**The impact of variance of reservation price distribution**

Another factor we investigate in our study is the variance of the reservation price distribution. We run our model for the equal standard deviation ($\sigma_1 = \sigma_2$) values of 1, 2 and 3 to investigate its effect. The other parameters are same as the base case ($\mu_1 = \mu_2 = 10$, $p_1 = p_2 = 10$, $Q_1 = Q_2 = 5$, $\lambda = 10$, $\rho = -0.9, 0, 0.9$, $c = 0, 1, 2, 4$). The results are tabulated in Table 5.

The obvious finding of this analysis is that the expected profit decreases as the variance of the reservation price distribution increases. For the same bundling cost value, the difference between the expected profit at $\sigma = 1$ and the expected profit at $\sigma = 3$ increases as the correlation coefficient increases. In other words, the effect of the variance of the reservation price distribution on the expected profit is much more pronounced when there is a positive correlation between the reservation prices. Figures 5, 6 and 7 demonstrate the effect of the standard deviation on the expected profit with correlation coefficient values of -0.9, 0 and 0.9, respectively. Again it confirms the previous findings that the expected profit decreases as the correlation coefficient increases. The optimal number of bundles decreases as the standard deviation increases. The optimal bundle price has different behavior with respect to the correlation coefficient. For negative correlation, the optimal bundle price decreases as the standard deviation increases. However for positive correlation, the optimal bundle price is an increasing function of the standard deviation. For the zero correlation case, the optimal bundle price is a decreasing function of the standard deviation for small bundling cost values and it is an increasing function for large bundling cost values.

**The impact of arrival rate**

Finally, we investigate the effect of the arrival rate, $\lambda$. The arrival rate has three different values (5, 10 and 15) for this setup. All other parameters are set as the same as the base case ($\mu_1 = \mu_2 = 10$, $\sigma_1 = \sigma_2 = 2$, $p_1 = p_2 = 10$, $Q_1 = Q_2 = 5$, $\lambda = 10$, $\rho = -0.9, 0, 0.9$, $c = 0, 1, 2, 4$). The results are tabulated in Table 6.

The clear finding of this case that can be seen from the Figures 8, 9 and 10 is that the expected profit is an increasing function of the arrival rate. The number of bundle sales decreases and the number of individual product sales increases when the arrival rate increases since the need for bundling decreases; the retailer can easily sell its products individually. For the negative correlation case, decrease in the optimal number of bundles formed is more
Figure 5  Profit vs. standard deviation for $\rho = -0.9$

Figure 6  Profit vs. standard deviation for $\rho = 0$

Figure 7  Profit vs. standard deviation for $\rho = 0.9$
significant. The retailer offers less bundles and charges higher prices for them as the arrival rate increases.

![Figure 8](image.png)

Figure 8  Profit vs. arrival rate for $\rho = -0.9$

![Figure 9](image.png)

Figure 9  Profit vs. arrival rate for $\rho = 0$

Substitutability and complementarity

We now investigate the effects of product substitutability and complementarity. The degree of contingency, $\theta$ considered in this section has five different values: -0.1, -0.05, 0, 0.05 and 0.1. The other parameters are same as the parameters in the base case. The results for the correlation coefficient, $\rho$, of -0.9, 0 and 0.9 are tabulated in Tables 7, 8 and 9 respectively. Figures 11, 12 and 13 depict the same results graphically.

Expected profit is an increasing function of degree of contingency under all correlation coefficient values. For the positive degree of contingency case (superadditive reservation prices),
customer willingness to purchase the bundle is higher than the negative degree of contingency case (subadditive reservation prices). Therefore for all correlation values and bundle formation costs, we see that the retailer is forming more bundles, or charging higher prices for the bundle or both as the degree of contingency increases. The highest jump in the profits occur when the degree of contingency increases to 0 from -0.05 which shows that the product substitutability has a significant impact on the efficiency of bundling and pricing. The impact of the degree of contingency is more pronounced when the product prices are uncorrelated or positively correlated. It seems that the retailer is already able to generate significant profits through bundling when the product reservation prices are negatively correlated and the impact of product complementarity is not significant.
4. Conclusion
We consider a retail firm that sells two types of perishable products in a single period not only as independent items but also as a bundle. Our emphasis is on understanding the bundling practices on the inventory and pricing decisions of the firm. We study the bundle formation and pricing problem of two products facing random demand, under inventory constraints over a finite selling horizon. After the retailer decides the number of bundles to be formed at the beginning of the season, no new bundles are formed and none of the bundles are unbundled to offer individual products during the season. Bundle formation costs are also included in the model.

Our numerical study shows that the optimal bundle price and expected profits are decreasing functions of the correlation coefficient. While the bundle formation cost has a significant impact on total profit, the impact on the number of bundles depends on the correlation coefficient. With negative correlation, bundling cost has a significant impact on the number of bundles. However, with positive correlation, this effect is negligible. When the individual
product prices are set below the mean reservation price, the retailer sets the highest possible bundle price and offers individual products as well as the bundle (mixed bundling). When the individual product prices are high, the retailer sets a bundle price such that only bundles are sold (pure bundling).

Other findings include the fact that the expected profit and the optimal number of bundles formed decreases as the variance of the reservation price distribution increases. The impact of the variance of the reservation price distribution on the expected profit is much more significant when there is a positive correlation between the reservation prices. We also perform analysis to investigate the product substitutability and complementarity. For the positive degree of contingency case (superadditive reservation prices), customer willingness to purchase bundle is higher than the negative degree of contingency case (subadditive reservation prices). Therefore for all correlation values and bundle formation costs, we see that the retailer forms more bundles, or charges higher prices for the bundle, or both as the degree of contingency increases. As a result, expected profit is an increasing function of degree of contingency.

As an important future research direction, the single period model in this paper can be extended to a multi-period model, allowing new bundle formations (and perhaps unbundling) and re-pricing at the beginning of each period. Also we can also extend the model to allow for replenishments of individual products at a certain cost. Another important but a complex extension of our work could be the modeling of competition.

References


Appendix 1: Purchasing and Switching Probabilities for $\theta \neq 0$

**Purchasing probabilities**

Remember that $m_0$ denotes the probability of no purchase, $m_1, m_2, m_B$ denote the probability of purchasing Product 1, Product 2 and the Bundle, respectively. Purchasing probabilities when we have a non–zero degree of contingency can be derived as follows.

\[
m_0 = P(R_1 < p_1; R_2 < p_2; R_b < p_b)
= \int_{p_1}^{p_2} \int_{r_1}^{r_2} f_{R_1,R_2}(r_1,r_2)dr_1dr_2
\]

\[
m_1 = P(R_1 \geq p_1; R_1 - p_1 \geq R_2 - p_2; R_1 - p_1 \geq R_b - p_b)
= \int_{p_1}^{p_2} \int_{r_1}^{p_2} f_{R_1,R_2}(r_1,r_2)dr_1dr_2
\]

\[
m_2 = P(R_2 \geq p_2; R_2 - p_2 \geq R_1 - p_1; R_2 - p_2 \geq R_b - p_b)
= \int_{p_2}^{\infty} \int_{r_1}^{p_2} f_{R_1,R_2}(r_1,r_2)dr_1dr_2
\]

\[
m_3 = P(R_b \geq p_b; R_b - p_b \geq R_1 - p_1; R_b - p_b \geq R_2 - p_2)
= \int_{p_b}^{\infty} \int_{r_1}^{\infty} f_{R_1,R_2}(r_1,r_2)dr_1dr_2
\]

where

\[
a_1 = \min \{p_2, (p_b - (1 + \theta)r_1)/(1 + \theta)\}
\]

\[
a_2 = \min \{r_1 - p_1 + p_2, (p_b - p_1 - \theta r_1)/(1 + \theta)\}
\]

\[
a_3 = \min \{r_2 - p_2 + p_1, (p_b - p_2 - \theta r_2)/(1 + \theta)\}
\]

\[
a_4 = \max \{(p_b - (1 + \theta)r_1)/(1 + \theta), (p_b - p_1 - \theta r_1)/(1 + \theta), (p_b - p_2 - (1 + \theta)r_1)/\theta\}
\]

**Switching probabilities**

*One type of product incurs shortage*

We first study the case when Product 1 runs out of stock and derive the expression for the probability of switching from Product 1 to the Bundle:

\[
\gamma_{1B} = \Pr \{R_2 - p_2 \geq R_2 - p_2; R_b \geq p_b; R_1 - p_1 \geq R_b - p_b; R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \}
\]

\[
= \Pr \left\{ \frac{(1 + \theta)(R_1 + R_2) - p_2}{R_1 + R_2} \geq \frac{(1 + \theta)(R_1 + R_2) - p_2}{R_1 + R_2}; R_1 - p_1 \geq R_1 - p_1 \right\} / m_1
\]

For $\theta > 0$, this expression can be written as:

\[
\gamma_{1B} = \Pr \left\{ R_2 \geq \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta}; R_2 \geq \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta} - R_1; R_2 \geq \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta} - R_1 \right\} / m_1
\]

\[
= \Pr \left\{ R_2 \geq \max \left\{ \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta}, \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta} \right\}; R_2 \leq \min \left\{ \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta}, R_1 - p_1 + p_2 \right\}; R_1 \geq p_1 \right\} / m_1
\]

\[
= \int_{p_1}^{\infty} \int_{\max \left\{ \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta}, \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta} \right\}}^{\min \left\{ \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta}, R_1 - p_1 + p_2 \right\}} f_{R_1,R_2}(r_1,r_2)dr_1dr_2/m_1
\]

For $\theta < 0$, this expression can be written as:

\[
\gamma_{1B} = \Pr \left\{ R_2 \leq \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta}; R_2 \geq \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta} - R_1; R_2 \geq \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta} - R_1 \right\} / m_1
\]

\[
= \Pr \left\{ R_2 \leq \min \left\{ \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta}, \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta} \right\}; R_2 \geq \max \left\{ \frac{p_2 - p_2 - (1 + \theta)R_1}{\theta}, R_1 - p_1 + p_2 \right\}; R_1 \geq p_1 \right\} / m_1
\]
for the case
Following similar arguments, \( \gamma_{12}, \gamma_{2B}, \gamma_{21}, \gamma_{B1} \) and \( \gamma_{B2} \) can be derived. The final expressions for the case \( \theta > 0 \) are given below:

\[
\gamma_{12} = \int_{p_1}^{\infty} \min_{p_2} \left( \frac{p_b - (1+\theta) r_1 - p_2}{\theta}, \frac{p_b - \theta r_1 - p_1}{\theta}, \frac{r_1 - p_1 + p_2}{\theta} \right) f_{R_1, R_2}(r_1, r_2) dr_1 dr_2/m_1
\]

\[
\gamma_{1B} = \int_{p_1}^{\infty} \min_{p_2} \left( \frac{p_b - (1+\theta) r_1 - p_2}{\theta}, \frac{p_b - \theta r_1 - p_1}{\theta}, \frac{r_1 - p_1 + p_2}{\theta} \right) f_{R_1, R_2}(r_1, r_2) dr_1 dr_2/m_1
\]

The final expressions for the case \( \theta < 0 \) are given below:

\[
\gamma_{12} = \int_{p_1}^{\infty} \max_{p_2} \left( \frac{p_b - (1+\theta) r_1 - p_1}{\theta}, \frac{p_b - (1+\theta) r_2 - p_2}{\theta}, \frac{r_1 - p_1 + p_2}{\theta} \right) f_{R_1, R_2}(r_1, r_2) dr_1 dr_2/m_1
\]

\[
\gamma_{1B} = \int_{p_1}^{\infty} \max_{p_2} \left( \frac{p_b - (1+\theta) r_1 - p_1}{\theta}, \frac{p_b - (1+\theta) r_2 - p_2}{\theta}, \frac{r_1 - p_1 + p_2}{\theta} \right) f_{R_1, R_2}(r_1, r_2) dr_1 dr_2/m_1
\]

Two types of products incur shortage
We first study the case when Product 1 and Product 2 run out of stock, and derive the expression for the switching probability from Product 1 to the Bundle.

\[
\gamma_{1B} = \Pr \{ R_b \geq p_b | R_1 - p_1 \geq R_b - p_b; R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \}
\]

\[
= \Pr \{ (1+\theta)(R_1 + R_2) \geq p_b; R_1 - p_1 \geq (1+\theta)(R_1 + R_2) - p_b; R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \} / m_1
\]

\[
= \Pr \{ R_2 \geq \frac{p_b}{1+\theta} - R_1; \frac{p_b - \theta R_1 - p_1}{1+\theta} \geq R_2; R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1 \} / m_1
\]

\[
= \Pr \{ R_2 \geq \frac{p_b}{1+\theta} - R_1; R_2 \leq \min \left( \frac{p_b - \theta R_1 - p_1}{1+\theta}, R_1 - p_1 + p_2 \right); R_1 \geq p_1 \} / m_1
\]

\[
= \int_{p_1}^{\infty} \min_{p_2} \left( \frac{p_b - \theta R_1 - p_1}{1+\theta}, R_1 - p_1 + p_2 \right) f_{R_1, R_2}(r_1, r_2) dr_1 dr_2/m_1
\]
Similarly expressions for $\gamma_{2B}, \gamma_{B1}, \gamma_{21}, \gamma_{B2}$, and $\gamma_{12}$ can be derived. Final expressions for $\gamma_{2B}, \gamma_{21},$ and $\gamma_{12}$ are given below:

$$\gamma_{2B} = \int_{p_2}^{\infty} \int_{p_1}^{\min\left(\frac{p_2 - \theta r_2 - p_2}{1+\theta}, r_2 - p_2 + p_1\right)} f_{R_1,R_2}(r_1,r_2) dr_2 dr_1 / m_2$$

$$\gamma_{21} = \int_{p_2}^{\infty} \int_{p_1}^{\min\left(\frac{p_2 - \theta r_2 - p_2}{1+\theta}, r_2 - p_2 + p_1\right)} f_{R_1,R_2}(r_1,r_2) dr_2 dr_1 / m_2$$

$$\gamma_{12} = \int_{p_1}^{\infty} \int_{p_2}^{\min\left(\frac{p_2 - \theta r_1 - p_1}{1+\theta}, r_1 - p_1 + p_2\right)} f_{R_1,R_2}(r_1,r_2) dr_1 dr_2 / m_1$$

Final expressions for $\gamma_{B1}$ and $\gamma_{B2}$ depend on the sign of $\theta$. These are given below for $\theta > 0$:

$$\gamma_{B1} = \int_{p_1}^{\infty} \int_{p_2}^{\max\left(\frac{p_2 - \theta r_1 - p_1}{1+\theta}, p_2 - \frac{(1+\theta)r_1 - p_2}{\theta}\right)} f_{R_1,R_2}(r_1,r_2) dr_2 dr_1 / m_b$$

$$\gamma_{B2} = \int_{p_2}^{\infty} \int_{p_1}^{\max\left(\frac{-r_2 - (1+\theta)r_2 + p_b}{1+\theta}, p_2 - \frac{\theta r_1 - p_1}{\theta}\right)} f_{R_1,R_2}(r_1,r_2) dr_1 dr_2 / m_b$$

and for $\theta < 0$:

$$\gamma_{B1} = \int_{p_1}^{\infty} \int_{p_2}^{\min\left(\frac{p_2 - \theta r_1 - p_2}{1+\theta}, p_2 - \frac{(1+\theta)r_1 - p_2}{\theta}\right)} f_{R_1,R_2}(r_1,r_2) dr_2 dr_1 / m_b$$

$$\gamma_{B2} = \int_{p_2}^{\infty} \int_{p_1}^{\min\left(\frac{p_2 - \theta r_2 - p_2}{1+\theta}, p_2 - \frac{(1+\theta)r_2 - p_2}{\theta}\right)} f_{R_1,R_2}(r_1,r_2) dr_1 dr_2 / m_b$$
Table 3 The impact of product prices on the expected profit: $\mu_1 = \mu_2 = 10, \sigma_1 = \sigma_2 = 2, \theta = 0, Q_1 = Q_2 = 5, \lambda = 10$

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Table 5  The impact of variance of reservation price distributions on the expected profit: $p_1 = p_2 = 10, \mu_1 = \mu_2 = 10, \theta = 0, Q_1 = Q_2 = 5, \lambda = 10$
Table 6 The impact of arrival rate on the expected profit: \( \lambda = 10 \), \( \mu = 10 \), \( \sigma_1 = \sigma_2 = 2 \), \( Q_1 = Q_2 = 5 \)

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Table 7 The impact of degree of contingency on the expected profit for \( \rho = -0.9 \), \( p_1 = p_2 = 10 \), \( \mu_1 = \mu_2 = 10 \), \( \sigma_1 = \sigma_2 = 2 \), \( Q_1 = Q_2 = 5 \), \( \lambda = 10 \)

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Table 8 The impact of degree of contingency on the expected profit for \( \rho = -0.9 \), \( p_1 = p_2 = 10 \), \( \mu_1 = \mu_2 = 10 \), \( \sigma_1 = \sigma_2 = 2 \), \( Q_1 = Q_2 = 5 \), \( \lambda = 10 \)
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\[
\text{Table 9} \quad \text{The impact of degree of contingency on the expected profit for } \rho = 0.9; \ p_1 = p_2 = 10, \ \mu_1 = \mu_2 = 10, \ \sigma_1 = \sigma_2 = 2, \ Q_1 = Q_2 = 5, \ \lambda = 10
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